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## Renormalization-group treatment of a pinning–depinning transition in an incommensurate structure

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**Abstract.** A new type of renormalization-group approach is suggested to treat the structure and properties of the Frenkel–Kontorova (discrete sine–Gordon) model ground states. Special attention is devoted to the incommensurate structures where the transition by breaking of analyticity of the hull function (pinning–depinning transition) occurs when the strength of the periodic potential is varied. The method gives for a particular incommensurability—the ‘golden mean’ one—a critical value of the potential strength one-and-a-half times larger than the exact value obtained by Aubry *et al* by numerical methods. The critical index of correlation length coincides within a few per cent accuracy with its exact value.

### 1. Introduction

There exist a number of physical systems where a great variety of periodically and quasi-periodically modulated structures may be realised as ground states. Examples are: charge-density waves in crystals with Peierls distortion [1–3], magnetic flux configurations in a periodically inhomogeneous Josephson junction [4], an adsorbed monolayer on a crystal surface [5, 6], vortex structures in a layered (e.g. high-temperature) superconductor [7], and magnetics with anisotropic competing interactions [8–10]. In each case the period of modulation is governed by some specific parameter—the density of electrons in a charge-density wave, the average magnetization in a Josephson junction and in a layered superconductor, the density of adatoms in a monolayer—or by its thermodynamically conjugate quantities: the chemical potential, the magnetic field and the pressure respectively.

The Frenkel–Kontorova (FK) model [11–13] serves as a standard tool to describe such systems [14, 15]. The usual way to obtain analytical results in this model was primarily the continuous approximation [12, 13], sometimes with variations such as e.g. continuous approximation for the  $M$ th-order commensurability [15]. Aubry *et al* [16, 17] have shown that one essential feature of the wide class of discrete models (including the FK one) with two competing length scales cannot be described properly in the continuous approximation. This is the transition between locked and unlocked configurations when the pinning strength parameter is varied. A sophisticated approach to study this transition is based on the analysis of the

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associated standard map [18, 19]. The most complete and accurate results for the FK model have recently been obtained in this way by MacKay with the renormalization of the standard map [18].

Here we suggest an attempt at a simple analytical approach to the problem based on the natural and physical picture of the ground state as some hierarchy of defects. This approach is the renormalization group (RG) of a type something like the real-space RG. Though within the framework of the RG approach in its present form only rough estimates of the critical quantities are possible, we believe it gives us somewhat deeper insight into the structure of incommensurate quasi-periodic states of discrete systems.

The formulation of the FK model and of its ground-state problem is given in section 2. The general description of various ground states in terms of Aubry's hull function [16, 17] is also presented. The qualitative description of the ground state as a hierarchy of defects is given in section 3. The idea of transition to a new set of variables—namely kink ones with the renormalized energy functional—is formulated in section 4. In section 5 we present the transition to the kink variables on the basis of the continuous approximation. The formulae for the RG transformations under various assumptions are obtained in section 6. The fixed cycles and points and the behaviour of the transformation in their neighbourhoods are discussed in section 7. The conclusions are formulated in section 8.

## 2. Formulation of the problem

The energy functional  $E(\{\phi_j\})$  for the FK chain may be written as follows:

$$E = \sum_j \left[ \frac{1}{2}(\phi_{j+1} - \phi_j)^2 + \lambda V(\phi_j) \right] \quad (2.1)$$

with

$$V(\phi) = [1/(4\pi^2)][1 - \cos(2\pi\phi)]. \quad (2.2)$$

Here the variables  $\phi_j$ ,  $-\infty < \phi_j < +\infty$ , are defined on all the sites of the chain with numbers  $j$ , and  $\lambda$  is the strength of the periodic potential. According to Aubry *et al* [16] let us set the ground-state problem in the following way: one should find the minimal energy configuration among those with some definite value of the 'mean phase difference'  $\Phi$ :

$$\Phi = \lim_{\substack{N \rightarrow +\infty \\ N' \rightarrow -\infty}} \frac{\phi_N - \phi_{N'}}{N - N'}. \quad (2.3)$$

In a wide class of discrete models including the FK one all the ground states with a given value of the mean phase difference  $\Phi$  may be written using the hull function [16]  $g(x, \Phi)$  as

$$\phi_j = g(j\Phi + \beta, \Phi). \quad (2.4)$$

The hull function  $g(x)$  ( $\Phi$  is treated as a parameter) has the following properties: (i) it is monotonically increasing,  $g(x_1) > g(x_2)$  if  $x_1 > x_2$ ; (ii)  $g(x+1) = g(x) + 1$ , or  $g(x) = x + h(x)$ , where  $h(x)$  is periodic with period 1. For the FK model

(2.1),  $g(x, \Phi)$  has two additional properties: (iii)  $g(x, \Phi + 1) = g(x, \Phi)$ ; and (iv)  $g(-x, \Phi) = -g(x, \Phi)$ . The first of the latter is the consequence of the quadraticity of the first term in the sum on the RHS of (2.1), and the second of the property  $V(\phi) = V(-\phi)$  in (2.2). The parameter  $\beta$  was introduced into (2.4) to describe the degeneracy of the ground state.

If  $\Phi$  is a rational number,  $\Phi = M/L$ , with  $M$  and  $L$  being integers, then  $g(x)$  is a discontinuous function with  $L$  jumps and  $L$  flat steps in between them within the unit interval of  $x$ . The ground state is  $L$ -fold degenerate commensurate in this case. Of course, there is no Goldstone mode and this state is always pinned.

The situation is different for incommensurate ground-state configurations with irrational values of the main phase difference  $\Phi$ . For small values of the potential strength  $\lambda$  one can prove using perturbation theory [17] that the hull function is a continuous analytic function of  $x$ . This is likely to correspond to the unpinned ground-state configurations. Indeed one such configuration may be continuously translated into another one by varying the degeneracy parameter  $\beta$ . On the other hand, if  $\lambda$  is large,  $\lambda \gg 1$ , the cosine potential (2.2) may be substituted by the piecewise-periodic parabolic one [20]. In the latter case the hull function can be found explicitly and appears to be a discontinuous one. The discontinuity points are located at  $x = l\Phi + m$ , where  $l$  and  $m$  are arbitrary integers; hence they are dense on the real axis. All the jumps are finite at the discontinuity points, and consequently the ground state is pinned.

### 3. The ground state as a hierarchy of defects

The simplest example of a kink-type defect is the well known sine-Gordon kink soliton. If  $\Phi = 0$  the ground-state configuration is simply  $\phi_j = n$ ,  $n$  being integer. The elementary excitation here is a *solitary kink*, i.e. the stationary configuration with the following properties:  $\phi_j \rightarrow n$  as  $j \rightarrow -\infty$ , and  $\phi_j \rightarrow n \pm 1$  as  $j \rightarrow +\infty$ .

The main difference between kinks in FK and sine-Gordon models is that, while in the latter case the centre of the kink may be situated at any point of the chain, in the former the kink centre should be in the middle between two sites and every two neighbouring positions are separated by the Peierls-Nabarro energy barriers [17].

The ground-state configuration for  $\Phi = 1/N$ , where  $N$  is integer, is a periodic one:

$$\phi_{j+N} = \phi_j + 1. \tag{3.1}$$

One can treat this configuration as a sequence of equidistant kinks with interkink distances equal to  $N$ . The configuration is  $N$ -fold degenerate. Strictly speaking, to describe the configuration as a soliton lattice one should suppose that the distance between kinks is larger than the kink length  $\sim \lambda^{-1/2}$ , i.e.  $\lambda^{1/2} N \gg 1$ . However, further on we shall use the soliton picture in a loose sense. The 'elementary phase defect' within the above-described ground-state configuration for  $\Phi = 1/N$ , or the second-generation soliton, is the stationary configuration  $\phi_j^{(d)}$  with the following property: there exists a ground-state configuration with  $\Phi = 1/N$ ,  $\phi_j^{(g)}$ , such that

$$\phi_j^{(d)} \rightarrow \begin{cases} \phi_j^{(g)} & \text{as } j \rightarrow -\infty \\ \phi_{j \pm 1}^{(g)} & \text{as } j \rightarrow +\infty \end{cases}. \tag{3.2}$$

This configuration can be loosely treated as a kink lattice with all the distances between neighbouring kinks equal to  $N$  except one that is equal to  $N \pm 1$ .

Let us consider the ground-state configuration for  $\Phi$ :

$$\Phi = 1/(N_1 \pm 1/N_2) = N_2/(N_1 N_2 \pm 1). \quad (3.3)$$

Here we have  $N_2$  kinks per period, i.e. per  $N_1 N_2 \pm 1$  sites. One can treat this configuration as an equidistant sequence of second-generation solitons. Here we have one such soliton for  $N_1 N_2 \pm 1$  sites or, equivalently, for  $N_2$  first-generation solitons (simple kinks).

Further generalization is obvious. For every commensurate ground state with mean phase difference  $\Phi^{(s)}$ ,

$$\Phi^{(s)} = M_{s+1}/L_{s+1} = 1/[N_1 \pm 1/(N_2 \pm \dots \pm 1/N_s) \dots] \quad (3.4)$$

where  $N \geq 2$  are naturals, we may introduce the elementary phase defect as follows: If  $\phi_j$  is the ground-state configuration with mean phase difference  $\Phi^{(s)}$ , then, in accordance with Aubry's theorem, it may be represented through the hull function  $g$  as  $\phi_j^{(s)} = g(j\Phi^{(s)} + \beta, \Phi^{(s)})$ . The defect configuration  $\phi_j^{(s,d)}$  is the minimal energy one with the property

$$\phi_j^{(s,d)} \rightarrow \begin{cases} g(j\Phi^{(s)} + \beta, \Phi^{(s)}) & \text{as } j \rightarrow -\infty \\ g(j\Phi^{(s)} + \beta \pm 1/L_{s+1}, \Phi^{(s)}) & \text{as } j \rightarrow +\infty \end{cases} \quad (3.5)$$

The minimal energy configuration is defined as a stationary one with the following property [16]: any change of any finite set of variables  $\phi_j$  necessarily increases energy (2.1).

Introducing one defect of (3.5) type per  $N_{s+1}$  periods into the  $\phi_j^{(s)}$  ground-state configuration, one obtains the  $(s+1)$ th-order ground-state configuration  $\phi_j^{(s+1)}$  with mean phase difference  $\Phi^{(s+1)}$ :

$$\Phi^{(s+1)} = M_{s+2}/L_{s+2} = 1/\{N_1 \pm 1/[N_2 \pm \dots \pm 1/(N_s \pm 1/N_{s+1}) \dots]\}. \quad (3.6)$$

Taking the limit  $s \rightarrow \infty$ , one can describe an incommensurate state as an infinite hierarchy of defects. The mean phase difference in this ground state is an irrational number  $\Phi$  represented by an infinite continuous fraction:

$$\Phi = \lim_{s \rightarrow \infty} \Phi^{(s)} = 1/[N_1 \pm 1/(N_2 \pm \dots) \dots]. \quad (3.7)$$

#### 4. Energy of the kink lattice

The energy functional (2.1) may be written as

$$E = \sum_j L_0(\phi_j, \phi_{j+1}). \quad (4.1)$$

In accordance with the concept set up in the previous section, the ground state of this functional under condition (2.3) with  $\Phi$  given by (3.4) or (3.7) can be viewed as a kink lattice with mean interkink distance equal to

$$1/\Phi \equiv 1/\Phi_0 = N_1 \pm \Phi_1 \quad (4.2)$$

where

$$\Phi_s = 1 / [N_{s+1} \pm 1 / (N_{s+2} \pm \dots) \dots] \quad 0 \leq \Phi_s \leq \frac{1}{2}. \quad (4.3)$$

The main idea of any renormalization group is in essence that of a partial reduction of degrees of freedom. Here we want to describe the configurations in the FK model in terms of new variables—the positions of kinks along the chain—instead of the initial phase variables  $\phi_j$ . Thus, we shall have  $\Phi L$  variables instead of  $L$  for the chain of  $L$  sites.

Therefore, we should invent in some way a new set of variables  $\psi_j$ , namely, kink coordinates (now  $j$  enumerates the kinks rather than the sites). Of course, the energy functional in terms of these new variables will contain not only the nearest-neighbour interactions but also the pair ones. Let us suppose, however, that, to find a ground-state configuration, it is possible to approximate the true energy functional by a pair one with nearest-neighbour interactions only:

$$E_{\text{eff}} = \sum_j \tilde{L}_1(\psi_j, \psi_{j+1}). \quad (4.4)$$

Let us consider the structure (not necessarily the stationary one) with all interkink distances being equal to some integer  $N$ :

$$\psi_1 = \psi + jN. \quad (4.5)$$

Its energy per kink (or per  $N$  sites of the initial lattice) is just  $\tilde{L}_1(\psi, \psi + N)$ . Note that it is not a ground-state energy for  $\Phi = 1/N$  but rather it corresponds to the ground state homogeneously shifted by  $\psi$ . In other words, the configuration of interest,  $\tilde{\phi}_j$ , should be the minimal energy configuration of (4.1) under the restrictions

$$\tilde{\phi}_{j+N} = \tilde{\phi}_j + 1 \quad (4.6)$$

$$\sum_{j=0}^{N-1} \tilde{\phi}_j = -\psi. \quad (4.7)$$

The condition (4.6) implies that the distances between neighbouring kinks are all equal to  $N$ . To explain the condition (4.7), let us note that the homogeneous shift of the kink chain for one lattice spacing to the right corresponds simply to the change of variables as  $\phi_j \rightarrow \phi_{j-1}$ , but after this equation (4.7) implies also the change  $\psi \rightarrow \psi + 1$ . Then the continuous change of  $\psi$  should result in the continuous homogeneous shift of kinks along the chain. The minus sign in (4.7) corresponds to the fact that the values of  $\phi_j$  are lowered when the kink chain is moving to the right. If the configuration  $\tilde{\phi}_j$  is determined, then

$$\tilde{L}_1(\psi, \psi + N) = \sum_{j=0}^{N-1} L_0(\tilde{\phi}_j, \tilde{\phi}_{j+1}). \quad (4.8)$$

Performing analytical continuation to the real values of  $N$  one obtains the kink pair energy  $\tilde{L}_1(\psi_0, \psi_1)$ .

Now we should minimize the new form of the energy functional (4.4) under the condition:

$$\langle \psi_{j+1} - \psi_j \rangle \equiv 1/\Phi_0 = N_1 + \Phi_1. \quad (4.9)$$

It is natural to introduce the variables  $\phi_j^{(1)}$ ,

$$\psi_j = N_1 j \pm \phi_j^{(1)} \quad (4.10)$$

with new mean phase difference  $\Phi_1$

$$\langle \phi_{j+1}^{(1)} - \phi_j^{(1)} \rangle = \Phi_1 \quad 0 \leq \Phi_1 \leq \frac{1}{2} \quad (4.11)$$

and new pair interaction  $\hat{L}_1$ ,

$$\hat{L}_1(\phi_j^{(1)}, \phi_{j+1}^{(1)}) = \bar{L}_1(\psi_j, \psi_{j+1}). \quad (4.12)$$

### 5. Continuous approximation

As a first step, we should minimize (4.1) under the conditions (4.6) and (4.7). To satisfy condition (4.7), one can add the term  $f\phi_j$  to  $L_0$  ( $f$  is the Lagrange variable) and minimize the energy functional with the pair interaction:

$$L_0 + f\phi_j = \frac{1}{2}(\phi_{j+1} - \phi_j)^2 + (\lambda/4\pi^2)[1 - \cos(2\pi\phi_j)] + f\phi_j. \quad (5.1)$$

The configuration of interest,  $\bar{\phi}_j$ , should satisfy the following stationarity equations:

$$2\bar{\phi}_j - \bar{\phi}_{j+1} - \bar{\phi}_{j-1} + (\lambda/2\pi) \sin(2\pi\bar{\phi}_j) + f = 0. \quad (5.2)$$

Equation (4.7) establishes the relation between  $f$  and  $\psi$ .

As  $\lambda \lesssim 1$ , the continuous approximation may be used to obtain the solution of equation (5.2). Replacing the discrete variable  $j$  by the continuous one  $x$  and  $\phi_{j+1} + \phi_{j-1} - 2\phi_j$  by the second derivative  $\phi''(x)$  transforms equation (5.2) into

$$\phi'' = (\lambda/2\pi) \sin(2\pi\phi) - f. \quad (5.3)$$

This equation has no homogeneous periodic solution satisfying equation (4.6), if  $f \neq 0$ . Hence  $f(\psi) = 0$  in the continuous approximation. Then the solution of equation (5.3) is

$$\phi(x) = \frac{1}{2} + (1/\pi) \text{am}((\lambda^{1/2}/k)(x - x_0), k) \quad (5.4)$$

where  $\text{am}(y, k)$  is the elliptic amplitude function with argument  $y$  and modulus  $x$ . The integration constants  $k$  and  $x_0$  should be adjusted to satisfy conditions (4.6) and (4.7).

The former case is satisfied if

$$2kK(k) = N\lambda^{1/2} \quad (5.5)$$

where  $K(k)$  is the complete elliptic integral of the first kind. As for condition (4.7), one can simply use  $x_0$  instead of  $\psi$ . Indeed, it is obvious that: (i)  $\psi(x_0)$  is an increasing function of  $x_0$ , and (ii)  $\psi(x_0 + 1) = \psi(x_0) + 1$ . Then one can use the  $x_0$  variable instead of  $\psi$ . Thus, further let us denote  $\psi = x_0$ .

Now we should evaluate sum (4.8) with  $\tilde{\phi}_j = \phi(j)$ , where  $\phi(x)$  is given by (5.4). The result is (see appendix)

$$\tilde{L}_1(\psi, \psi + N) = \mathcal{E}_0(N) - \sum_{m=1}^{\infty} D_m(N) \cos(2\pi m\psi) \tag{5.6}$$

where

$$\mathcal{E}_0(N) = \frac{1}{2N} + \frac{2}{N} \sum_{n=1}^{\infty} \left[ \frac{1}{1 + \cosh[(2n - 1)\alpha]} + \left( \frac{\sin(\pi n/N)}{\pi n/N} \right)^2 \frac{1}{1 + \cosh(2n\alpha)} \right] \tag{5.7}$$

$$D_m(N) = \frac{2m}{\sinh(mN\alpha)} - \frac{4}{\pi^2 m} \sum_{n=-\infty}^{\infty} \frac{\sin^2(\pi n/N)}{n} \times [1/\{\cosh(mN\alpha) + \cosh[(mN - 2n)\alpha]\}] \tag{5.8}$$

$$\alpha = \pi K'/K. \tag{5.9}$$

Equations (5.6)–(5.9) were obtained in the next-to-continuous approximation, i.e. the condition  $\lambda \ll 1$  was used. They may be simplified in the limiting cases  $N\lambda^{1/2} \ll 1$  and  $N\lambda^{1/2} \gg 1$ . In the former case equation (5.5) may be solved approximately as

$$K \simeq \pi/2 \quad k \simeq N\lambda^{1/2}/\pi \quad K' = K(k') \simeq \log(4/k) \simeq \log(4\pi/N\lambda^{1/2}). \tag{5.10}$$

Substituting (5.10) into (5.9) we have

$$\alpha \simeq \log(16\pi^2/\lambda N^2) \gg 1. \tag{5.11}$$

Then equations (5.7) and (5.8) are transformed to

$$\mathcal{E}_0(N) \simeq \frac{1}{2}N \tag{5.12}$$

$$D_m(N) \simeq A_m(N)(\lambda N^2/16\pi^2)^{mN} \tag{5.13}$$

$$A_m(N) = 4m - \frac{8}{\pi^2 m} \sum_{n=1}^{mN-1} \frac{\sin^2(\pi n/N)}{n}. \tag{5.14}$$

In the opposite case  $N\lambda^{1/2} \gg 1$  the solution of equation (5.5) is

$$K \simeq N\lambda^{1/2}/2 \quad k \simeq 1 \quad K' \simeq \pi/2. \tag{5.15}$$

Here we have

$$\alpha \simeq \pi^2/N\lambda^{1/2} \ll 1 \tag{5.16}$$

but

$$N\alpha \simeq \pi^2/\lambda^{1/2} \gg 1. \tag{5.17}$$



Owing to condition (5.16) one can replace the sums in (5.7) and (5.8) by integrals. The latter can be easily evaluated taking into account condition (5.17). Finally we have

$$\mathcal{E}_0(N) \simeq 2\lambda^{1/2}/\pi^2 + (8\lambda^{1/2}/\pi^2) \exp(-N\lambda^{1/2}) \quad (5.18)$$

$$D_m(N) \simeq A_m(\infty) \exp(-\pi^2 m/\lambda^{1/2}) \quad (5.19)$$

with

$$A_m(\infty) = \lim_{N \rightarrow \infty} A_m(N) = 4m - \frac{8}{\pi^2 m} \int_0^{\pi m} \frac{dx}{x} \sin^2 x. \quad (5.20)$$

One can conclude from (5.13) and (5.19) that, assuming  $\lambda \ll 1$  (a necessary condition to use the continuous approximation), all the terms except the first one with  $m = 1$  may be abandoned in (5.6).

## 6. Renormalization transformation

To obtain the pair kink interaction  $\bar{L}_1(\psi_1, \psi_2)$  (including the interaction with the lattice), one should perform analytic continuation on the  $N = \psi_1 - \psi_2$  variable from its integer values to the real axis. Taking into account the considerations given at the end of the previous section, we have

$$\bar{L}_1(\psi_1, \psi_2) = \mathcal{E}_0(\psi_1 - \psi_2) - D_1(\psi_1 - \psi_2) \cos(2\pi\psi_1). \quad (6.1)$$

If we restrict ourselves to the case of small lattice-kinks interaction  $D_1$ , then it is possible to replace

$$\psi_{j+1} - \psi_j \rightarrow \langle \psi_{j+1} - \psi_j \rangle \equiv 1/\Phi_0 \quad (6.2)$$

in the argument of  $D_1$ . Moreover, one can expand  $\mathcal{E}_0(\psi_{j+1} - \psi_j)$  near the mean value  $1/\Phi_0$ , restricting the expansion to the quadratic term. Then one may reproduce the initial form (2.1) of the energy functional after its renormalization in order to set the coefficient before the quadratic term in the expansion of  $\mathcal{E}_0$  equal to  $\frac{1}{2}$ . Omitting the irrelevant terms from the constant and linear parts of the  $\mathcal{E}_0$  expansion and introducing the new variables  $\phi_j^{(1)}$  through (4.10), we finally have

$$E_1(\{\phi_j^{(1)}\}) = [\mathcal{E}_0''(1/\Phi_0)]^{-1} E_0 = \sum_j L_1(\phi_j^{(1)}, \phi_{j+1}^{(1)}) \quad (6.3)$$

$$L_1(\phi_1, \phi_2) = \frac{1}{2}(\phi_2 - \phi_1)^2 + (\lambda_1/4\pi^2)[1 - \cos(2\pi\phi_1)] \quad (6.4)$$

where

$$\lambda_1 = 4\pi^2 |D_1(\Phi_0^{-1})/\mathcal{E}_0''(\Phi_0^{-1})|. \quad (6.5)$$

Here  $D_1(\Phi_0^{-1})$  and  $\mathcal{E}_0(\Phi_0^{-1})$  are defined through the analytic continuation of equations (5.1) and (5.8) respectively. If  $\lambda^{1/2} \ll \Phi$ , using equations (5.12) and (5.13), one may obtain the following expression for  $\lambda_1$ :

$$\lambda_1 = (4\pi^2/\Phi_0^3) A_1(1/\Phi_0) (\lambda_0/16\pi^2 \Phi_0^3)^{1/\Phi_0}. \quad (6.6)$$

In the opposite case  $1 \gg \lambda_0^{1/2} \gg \Phi_0$  we have from equations (5.18), (5.19) and (6.5):

$$\lambda_1 \simeq (\pi^4/2\lambda_0^{3/2})A_1(\infty) \exp(\lambda_0^{1/2}/\Phi_0 - \pi^2/\lambda_0^{1/2}). \tag{6.7}$$

The problem of the ground state of the energy functional (2.1) contains two independent parameters,  $\lambda_0$  and  $\Phi_0$ . Using the technique developed here, one can transform the initial values of these parameters into the equivalent ones  $\lambda_1$  and  $\Phi_1$  with  $\lambda_1(\lambda_0, \Phi_0)$  given by equation (6.5) or by equations (6.6) or (6.7) in the corresponding limiting cases. The new mean phase difference  $\Phi_1$  is given by equation (4.14), or

$$\Phi_1 = |1/\Phi_0 - \text{Int}(1/\Phi_0 + \frac{1}{2})| \tag{6.8}$$

where  $\text{Int } x$  denotes the integer part of  $x$ .

### 7. Fixed cycles and fixed points

The critical values  $\lambda_c(\Phi)$  correspond to fixed points or fixed cycles of the renormalization transformation in  $(\lambda, \Phi)$  space. Note that the sequence of transformation  $\Phi_j(\Phi_{j-1})$  (6.8) corresponds to continuous fraction expansion (3.7) of the initial value of the mean phase difference  $\Phi_0 \equiv \Phi$ . After  $s$  steps of the renormalization transformation we obtain the new mean phase difference equal to the  $s$ th-order remainder (4.3) of the continuous fraction (3.7).

To organize the cyclic sequence  $\Phi_s$ , one should initially have a periodic continuous fraction  $\Phi_0$ , i.e. the one with the periodic sequence of denominators  $N_s$ . It is well known that such a continuous fraction represents a quadratic irrationality [21], i.e. the root of a quadratic equation with integer coefficients. Hence explicit results about the values of  $\lambda_c$  and the critical behaviour of the system are available within the framework of the renormalization group suggested here for quadratic irrationalities only.

The simplest case of the fixed cycle is the fixed point. It corresponds to the value of the mean phase difference  $\Phi_N^\pm$  satisfying the equation

$$\Phi_N^\pm = 1/(N \pm \Phi_N^\pm) \tag{7.1}$$

where  $N$  is integer,  $N \geq 2$  for the plus sign and  $N > 2$  for the minus. Then  $\lambda_0(\Phi)$  may be obtained from equation (6.5) with  $\lambda_0 = \lambda_1 = \lambda_c$ . Using the simplified forms (6.6) or (6.7) of equation (6.5), one obtains for  $\lambda_c$ , respectively, either

$$\lambda_c(\Phi) = 16\pi^2\Phi^{(2+3\Phi)/(1-\Phi)}/[A_1(1/\Phi)/4]^{2/(1-\Phi)} \tag{7.2}$$

or

$$\lambda_c(\Phi) = \pi^2\Phi - \pi\Phi^{3/2} \log[A_1(1/\Phi)/2\pi\Phi^{5/2}]. \tag{7.3}$$

This fixed point is an unstable one. For  $\lambda_0 > \lambda_c$  the sequence  $\lambda_s \rightarrow \infty$ . This means a pinned ground-state configuration. If  $\lambda < \lambda_c$ , then  $\lambda_s \rightarrow 0$  and the ground state is unpinned.

To compare the results of our renormalization group and the exact ones obtained by straightforward numerical calculations, let us choose  $\Phi = 1 - G$ , where  $G = (\sqrt{5} - 1)/2$  is the golden mean value. Owing to the quadratic nature of the pair

interaction in (2.1),  $\lambda_c(\Phi) = \lambda_c(G)$ .  $\Phi$  should obey equation (7.1) with  $N = 3$  and the minus sign, i.e.

$$\Phi = (3 - \sqrt{5})/2 = 1/(3 - 1/(3 - \dots)) = \Phi_3^-. \quad (7.4)$$

Substituting (7.4) into (7.2), one obtains  $\lambda_c \simeq 1.32$ . The numerical methods give a critical value about one-and-a-half times lower,  $\lambda_c \simeq 0.902$  [22] (let us note that our definition of  $\lambda$  differs from Aubry's by the factor of  $\pi^2/2$ ).

The method suggested here is somewhat similar to the real-space renormalization-group approach [23]. The transition from initial phase variables to kink coordinates means that one retains some part of the initial degrees of freedom whose number is related to the number of the initial ones as  $\Phi$ . In other words, the length scale increases as  $1/\Phi$ . The index of correlation length (i.e. the inverse of Lyapunov exponent) is defined through the behaviour of  $\lambda_1(\lambda_0)$  transformation in the vicinity of  $\lambda_c$  as [24]:

$$\nu = \log(1/\Phi) / \log(\partial \lambda_1 / \partial \lambda_0)|_{\lambda_0 = \lambda_c} = 1. \quad (7.5)$$

Here we take the  $\lambda_1(\lambda_0)$  dependence from the simplified formula (6.6). The numerical methods give the value  $\nu = 0.9874625$  [20], i.e. a value very close to unity. The value  $\nu = 1$  appears due to the dependence

$$\lambda_{s+1} \sim \lambda_s^{1/\Phi}. \quad (7.6)$$

in approximate formula (6.6).

## 8. Conclusions

An approach suggested here seems to be somewhat qualitative, rather unsuitable for obtaining exact quantitative results about critical points and critical behaviour in Frenkel-Kontorova-like models. Surprisingly, the discrepancy between our renormalization-group results and the ones of numerical calculations appeared to be not as large as one may expect taking into account the very coarse character of uncontrollable approximations we have used here. The most doubtful is assumption (4.6) about the pair interaction of nearest-neighbour kinks only in the effective free-energy functional. The only intuitive arguments we can suggest in defence of this assumption is that systems with a finite range of interaction (the range of the kink interaction equal to  $\lambda^{-1/2}$  in the FK model) are governed on the macroscopic scale by two main parameters: the effective compressibility and the parameter of 'substrate potential'. The latter may be extracted from the pinning strengths of the integer commensurate configurations, where the pinning force is maximal relative to the other neighbouring ground-state ones. Thus, the only relevant quantity should be the dimensionless (i.e. related to the compressibility) coupling to the periodic potential.

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**Appendix**

Substituting  $\bar{\phi}_j = \phi(j)$  from (5.4) into (5.1) and then into (4.8) one obtains

$$\begin{aligned} \bar{L}_1(\psi, \psi + N) = \sum_{j=0}^{N-1} \left\{ \frac{1}{2\pi^2} \left[ \operatorname{am} \left( \frac{\lambda^{1/2}}{k} (j - \psi + 1) \right) \right. \right. \\ \left. \left. - \operatorname{am} \left( \frac{\lambda^{1/2}}{k} (j - \psi) \right) \right]^2 + \frac{\lambda}{2\pi^2} \operatorname{cn}^2 \left( \frac{\lambda^{1/2}}{k} (j - \psi) \right) \right\} \end{aligned} \tag{A1}$$

where  $\operatorname{cn}$  is the Jacobi elliptic cosine function. The following series expansions are useful [25]:

$$\operatorname{am} \left( \frac{\lambda^{1/2}}{k} x \right) = \frac{1}{2} \sum_n \frac{1}{n} \frac{\sin(2\pi n x/N)}{\cosh(n\alpha)} \tag{A2}$$

$$\operatorname{cn} \left( \frac{\lambda^{1/2}}{k} x \right) = \frac{\pi}{N\lambda^{1/2}} \sum_n \frac{\cos[2\pi(n - 1/2)x/N]}{\cosh[(n - 1/2)\alpha]} \tag{A3}$$

The undefined limits of summation mean the ones from  $-\infty$  to  $+\infty$ .  $\alpha$  is given by equation (5.9). Relation (5.5) has been taken into account in equations (A2) and (A3). Substituting (A2) and (A3) into (A1), we have

$$\begin{aligned} L_1(\psi, \psi + N) = \frac{1}{N^2} \sum_{j=0}^{N-1} \sum_{n,l} \left( \frac{\sin(\pi n/N)}{\pi n/N} \frac{\sin(\pi l/N)}{\pi l/N} \right. \\ \times \frac{\cos[2\pi(n - l)(j - \psi + 1/2)/N]}{\cosh[(n - l)\alpha] + \cosh[(n + l)\alpha]} \\ \left. + \frac{\cos[2\pi(n - l)(j - \psi)/N]}{\cosh[(n - l)\alpha] + \cosh[(n + l - 1)\alpha]} \right). \end{aligned} \tag{A4}$$

After the summation on  $j$  only terms with  $n - l = mN$ ,  $m$  being integer, survive. Then after some simple transformations,

$$\begin{aligned} \bar{L}_1(\psi, \psi + N) = \frac{1}{N} \sum_n \left[ \frac{1}{1 + \cosh[(2n - 1)\alpha]} + \left( \frac{\sin(\pi n/N)}{\pi n/N} \right)^2 \frac{1}{1 + \cosh(2n\alpha)} \right] \\ + \frac{2}{N} \sum_{m=1}^{\infty} \cos(2\pi m\psi) \sum_n \left( \frac{1}{\cosh(mN\alpha) + \cosh[(mN - 2n + 1)\alpha]} \right. \\ \left. - \frac{1}{\pi m} \frac{\sin^2(\pi n/N)}{\pi n/N} \frac{1}{\cosh(mN\alpha) + \cosh[(mN - 2n)\alpha]} \right). \end{aligned} \tag{A5}$$

Taking into account the following formula:

$$\sum_n \frac{1}{\cosh(mN\alpha) + \cosh[(mN - 2n + 1)\alpha]} = \frac{mN}{\sinh(mN\alpha)} \tag{A6}$$

one can easily approach equations (5.6)–(5.8).

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